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Quantum Entanglement in Two-Photon Tavis–Cummings Model with a Kerr Nonlinearity

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Abstract The properties of quantum entanglement in the two-photon Tavis–Cummings model with a Kerr nonlinearity are studied in terms of quantum information entropy theory. The reduced quantum entropy is employed to investigate the quantum entanglement between two two-level atoms and a single-mode coherent field. The relative quantum entropy is employed to investigate the quantum entanglement between the two two-level atoms. The influences of the nonlinear interaction of the Kerr medium with the field and the atomic dipole-dipole interaction on the properties of quantum entanglement of the system are also examined. Some important results are obtained.

Keywords Quantum entanglement · Reduced quantum entropy · Relative quantum entropy

1 Introduction

In recent years much attention has been focused on the properties of quantum entanglement between the field and the atom due to the entropy theory of the interaction of the field with the atom presented by Phoenix and Knight (PK) [13, 14]. The PK entropy theory tells us that the reduced quantum entropy is a very accurate measure of the degree of the entanglement between two components. The higher the reduced quantum entropy, the greater the entanglement. On the other hand, quantum entanglement is one of the most profound features of quantum mechanics and has been considered to be a valuable physical resource in the young field of quantum information science, including quantum computation [6], quantum teleportation [3], quantum dense coding [5], and quantum information processing [4]. Therefore, the study for quantum entanglement is very significant.

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Some authors have investigated the properties of the quantum entanglement in the J-C model and its other forms extended [1, 8–12]. However, less attention has been paid to the properties of quantum entanglement in the two-photon Tavis–Cummings model with a Kerr nonlinearity. In this paper, we will investigate the properties of quantum entanglement in the two-photon Tavis–Cummings model with a Kerr nonlinearity according to quantum information entropy theory and examine the influences of the nonlinear interaction of the Kerr medium with the field and the atomic dipole-dipole interaction on the properties of quantum entanglement of the system.

2 The Model and Its Solution

The system under consideration consists of two two-level atoms and a single-mode coherent field. And the Hamiltonian of this system in the rotating-wave approximation is [7]

$$H = \omega a^+ a + \omega_0 \sum_{l=1}^{2} S_3^{(l)} + g \sum_{l=1}^{2} (a^2 S_+^{(l)} + a^{+2} S_-^{(l)}) + \Omega (S_+^{(1)} S_-^{(2)} + S_-^{(1)} S_+^{(2)}) + \chi (a^+ a)^2. \quad (1)$$

Where $a^+(a)$ is creation (annihilation) operator for the single-mode coherent field, $S_3^{(l)}$ and $S_\pm^{(l)}$ are the pseudospin operators of the lth atom, ω and ω_0 are the frequencies of the field and the atomic transition respectively, g is the atoms-field coupling constant, Ω is the atomic dipole-dipole interacting constant, χ is the Kerr coupling constant. For simplicity, we only consider the resonant case ($\omega = \omega_0$).

We consider that at $t = 0$ the two atoms are in the excited state $|\psi_A(0)\rangle = |{+}{+}\rangle$ and the field is coherent

$$|\psi_F(0)\rangle = \sum_{n=0}^{\infty} f_n |n\rangle, \quad (2)$$

where

$$f_n = \exp(-N/2) \frac{N^{n/2}}{\sqrt{n!}} \exp(in\varphi). \quad (3)$$

Where N is the average photon number of the coherent field, and φ is the phase angle of the coherent field (in this paper we take $\varphi = 0$). The initial state of the system is a decoupled pure state, and the state vector is

$$|\psi_{FA}(0)\rangle = |\psi_F(0)\rangle \otimes |\psi_A(0)\rangle. \quad (4)$$

By solving the Schrodinger equation of the system in the interaction picture, we can obtain the system vector at time t

$$|\psi_{FA}(t)\rangle = |a\rangle|{+}{+}\rangle + |b\rangle|{+}{-}\rangle + |c\rangle|{-}{+}\rangle + |d\rangle|{-}{-}\rangle, \quad (5)$$

with

$$|a\rangle = \sum_{n=0}^{\infty} a(n,t)|n\rangle, \quad (6a)$$

$$|b\rangle = \sum_{n=0}^{\infty} b(n,t)|n+2\rangle, \tag{6b}$$

$$|c\rangle = \sum_{n=0}^{\infty} c(n,t)|n+2\rangle, \tag{6c}$$

$$|d\rangle = \sum_{n=0}^{\infty} d(n,t)|n+4\rangle, \tag{6d}$$

where

$$a(n,t) = k_n e^{i\alpha_n t} + \lambda_n e^{i\beta_n t} + \eta_n e^{i\gamma_n t}, \tag{7a}$$

$$\begin{aligned} b(n,t) &= c(n,t) \\ &= -\frac{k_n(\alpha_n + n^2\chi)}{2\sqrt{(n+2)(n+1)}g} e^{i\alpha_n t} - \frac{\lambda_n(\beta_n + n^2\chi)}{2\sqrt{(n+2)(n+1)}g} e^{i\beta_n t} \\ &\quad - \frac{\eta_n(\gamma_n + n^2\chi)}{2\sqrt{(n+2)(n+1)}g} e^{i\gamma_n t}, \end{aligned} \tag{7b}$$

$$\begin{aligned} &d(n,t) \\ &= \frac{k_n\{\alpha_n^2 + \alpha_n[2(n^2+2n+2)\chi + \Omega] - 2(n^2+3n+2)g^2 + n^2\chi[(n+2)^2\chi + \Omega]\}}{2\sqrt{(n+4)(n+3)(n+2)(n+1)}g^2} e^{i\alpha_n t} \\ &+ \frac{\lambda_n\{\beta_n^2 + \beta_n[2(n^2+2n+2)\chi + \Omega] - 2(n^2+3n+2)g^2 + n^2\chi[(n+2)^2\chi + \Omega]\}}{2\sqrt{(n+4)(n+3)(n+2)(n+1)}g^2} \\ &\times e^{i\beta_n t} \\ &+ \frac{\eta_n\{\gamma_n^2 + \gamma_n[2(n^2+2n+2)\chi + \Omega] - 2(n^2+3n+2)g^2 + n^2\chi[(n+2)^2\chi + \Omega]\}}{2\sqrt{(n+4)(n+3)(n+2)(n+1)}g^2} \\ &\times e^{i\gamma_n t}, \end{aligned} \tag{7c}$$

and

$$k_n = \frac{2(n^2+3n+2)g^2 + n^2\chi(n^2\chi + \gamma_n) + \beta_n(n^2\chi + \gamma_n)}{(\alpha_n - \beta_n)(\alpha_n - \gamma_n)} f_n, \tag{8a}$$

$$\lambda_n = \frac{2(n^2+3n+2)g^2 + n^2\chi(n^2\chi + \gamma_n) + \alpha_n(n^2\chi + \gamma_n)}{(\beta_n - \alpha_n)(\beta_n - \gamma_n)} f_n, \tag{8b}$$

$$\eta_n = \frac{2(n^2+3n+2)g^2 + n^2\chi(n^2\chi + \beta_n) + \alpha_n(n^2\chi + \beta_n)}{(\gamma_n - \alpha_n)(\gamma_n - \beta_n)} f_n, \tag{8c}$$

$$\alpha_n = -\frac{p_n}{3} - \frac{2^{1/3}h_n}{3v_n} + \frac{v_n}{3\times 2^{1/3}}, \tag{8d}$$

$$\beta_n = -\frac{p_n}{3} + \frac{(1+i\sqrt{3})h_n}{3\times 2^{2/3}v_n} - \frac{(1-i\sqrt{3})v_n}{6\times 2^{1/3}}, \tag{8e}$$

$$\gamma_n = -\frac{p_n}{3} + \frac{(1-i\sqrt{3})h_n}{3\times 2^{2/3}v_n} - \frac{(1+i\sqrt{3})v_n}{6\times 2^{1/3}}, \tag{8f}$$

$$h_n = -p_n^2 + 3q_n, \tag{8g}$$

$$v_n = \left[-2p_n^3 + 9p_nq_n + \sqrt{4(-p_n^2 + 3q_n)^3 + (-2p_n^3 + 9p_nq_n - 27w_n)^2} - 27w_n\right]^{1/3}, \quad (8h)$$

$$p_n = (3n^2 + 12n + 20)\chi + \Omega, \quad (8i)$$

$$q_n = -4(n^2 + 5n + 7)g^2 + \chi[(3n^4 + 24n^3 + 72n^2 + 96n + 64)\chi + 2(n^2 + 4n + 8)\Omega], \quad (8j)$$

$$w_n = (n + 4)\chi\{-4(n^3 + 5n^2 + 7n + 4)g^2 + n^2(n + 4)\chi[(n + 2)^2\chi + \Omega]\}. \quad (8k)$$

At any time $t > 0$ the reduced two atomic density operator for the system is given by

$$\rho_A(t) = \mathop{\mathrm{Tr}}_F |\psi_{FA}(t)\rangle\langle\psi_{FA}(t)| = \begin{bmatrix} \langle a|a\rangle & \langle b|a\rangle & \langle c|a\rangle & \langle d|a\rangle \\ \langle a|b\rangle & \langle b|b\rangle & \langle c|b\rangle & \langle d|b\rangle \\ \langle a|c\rangle & \langle b|c\rangle & \langle c|c\rangle & \langle d|c\rangle \\ \langle a|d\rangle & \langle b|d\rangle & \langle c|d\rangle & \langle d|d\rangle \end{bmatrix}. \quad (9)$$

The eigenvalues of the reduced two atomic density operator $\rho_A(t)$ can be obtained as follows:

$$\lambda_A^1 = 0, \quad (10a)$$

$$\lambda_A^2 = \frac{1}{6}\left[2 - \frac{2 \times 2^{1/3}(3A - 1)}{\varepsilon} + 2^{2/3}\varepsilon\right], \quad (10b)$$

$$\lambda_A^3 = \frac{1}{6}\left[2 + \frac{2(-2)^{1/3}(3A - 1)}{\varepsilon} + (-2)^{2/3}\varepsilon\right], \quad (10c)$$

$$\lambda_A^4 = \frac{1}{6}\left[2 - \frac{2(-1)^{2/3}2^{1/3}(3A - 1)}{\varepsilon} - (-1)^{1/3}2^{2/3}\varepsilon\right], \quad (10d)$$

where

$$\varepsilon = \left[2 - 9A - 27B + \sqrt{4(3A - 1)^3 + (-2 + 9A + 27B)^2}\right]^{1/3}, \quad (11)$$

$$A = \langle a|a\rangle\langle b|b\rangle + \langle a|a\rangle\langle c|c\rangle + \langle a|a\rangle\langle d|d\rangle + \langle b|b\rangle\langle c|c\rangle + \langle b|b\rangle\langle d|d\rangle + \langle c|c\rangle\langle d|d\rangle - \langle b|a\rangle\langle a|b\rangle - \langle c|a\rangle\langle a|c\rangle - \langle d|a\rangle\langle a|d\rangle - \langle c|b\rangle\langle b|c\rangle - \langle d|b\rangle\langle b|d\rangle - \langle d|c\rangle\langle c|d\rangle, \quad (12)$$

$$\begin{aligned} B = {} & -\langle a|a\rangle\langle b|b\rangle\langle c|c\rangle - \langle a|a\rangle\langle b|b\rangle\langle d|d\rangle - \langle a|a\rangle\langle c|c\rangle\langle d|d\rangle - \langle b|b\rangle\langle c|c\rangle\langle d|d\rangle \\ & + \langle a|a\rangle\langle c|b\rangle\langle b|c\rangle + \langle a|a\rangle\langle d|b\rangle\langle b|d\rangle + \langle a|a\rangle\langle d|c\rangle\langle c|d\rangle + \langle b|b\rangle\langle c|a\rangle\langle a|c\rangle \\ & + \langle b|b\rangle\langle d|a\rangle\langle a|d\rangle + \langle b|b\rangle\langle d|c\rangle\langle c|d\rangle + \langle c|c\rangle\langle b|a\rangle\langle a|b\rangle + \langle c|c\rangle\langle d|a\rangle\langle a|d\rangle \\ & + \langle c|c\rangle\langle d|b\rangle\langle b|d\rangle + \langle d|d\rangle\langle b|a\rangle\langle a|b\rangle + \langle d|d\rangle\langle c|a\rangle\langle a|c\rangle + \langle d|d\rangle\langle c|b\rangle\langle b|c\rangle \\ & - \langle b|a\rangle\langle c|b\rangle\langle a|c\rangle - \langle b|a\rangle\langle d|b\rangle\langle a|d\rangle - \langle c|a\rangle\langle a|b\rangle\langle b|c\rangle - \langle c|a\rangle\langle d|c\rangle\langle a|d\rangle \\ & - \langle d|a\rangle\langle a|b\rangle\langle b|d\rangle - \langle d|a\rangle\langle a|c\rangle\langle c|d\rangle - \langle c|b\rangle\langle d|c\rangle\langle b|d\rangle - \langle d|b\rangle\langle b|c\rangle\langle c|d\rangle. \end{aligned} \quad (13)$$

3 The Quantum Entanglement Between the Two Two-Level Atoms and the Coherent Field

We use the reduced quantum entropy as the measure of the degree of quantum entanglement between two two-level atoms and the coherent field. The entropy of the subsystem can be defined through their respective reduced density matrix as [16]

$$S_i(t) = -\operatorname*{Tr}_i[\rho_i(t)\ln\rho_i(t)] \quad (i = A \text{ or } F). \tag{14}$$

Since we have assumed that the two atoms and the coherent field are initially in a disentangled pure state, the total entropy of the system is zero. In terms of the triangle inequality of the entropy [2]

$$|S_A(t) - S_F(t)| \le S_{FA}(t) \le |S_A(t) + S_F(t)|, \tag{15}$$

we can see that the reduced entropies of the two subsystems are identical, namely $S_A(t) = S_F(t)$. In (15), $S_{FA}(t)$ is the total entropy of the system. Consequently we only need to calculate the atomic reduced entropy $S_A(t)$. We can express the reduced entropy $S_A(t)$ of the two atoms in terms of the eigenvalues of $\rho_A(t)$ given by (9),

$$S_A(t) = -\sum_{i=1}^{4}\lambda_A^i \ln\lambda_A^i. \tag{16}$$

It reflects the degree of the entanglement between the two atoms and the coherent field (DEAF). If $S_A(t)$ takes its minimal value zero, the two atoms and the field are disentangled. If $S_A(t)$ takes its nonzero value, the two atoms and the field are entangled.

Figure 1 displays the numerical results for the time evolution of the DEAF for $N = 10$, $\chi/g = 1$ and the different atomic dipole-dipole interaction intensities ($\Omega/g = 0, 25, 50$). When the atomic dipole-dipole interaction is equal to zero ($\Omega/g = 0$), the time evolution of the DEAF exhibits periodic oscillation and the two atoms are periodically disentangled from the coherent field. With the intensity of the atomic dipole-dipole interaction increasing, the time evolution of the DEAF exhibits irregular oscillation and the maximal value of the DEAF gradually decreases.

Figure 2 displays the numerical results for the time evolution of the DEAF for $N = 10$, $\Omega/g = 10$ and the different nonlinear interaction of the Kerr medium with the field ($\chi/g = 0.5, 1, 5$). From Figs. 2(a–c), it is observed that the DEAF evolves periodically. With the increase of the nonlinear interaction of the Kerr medium with the field, the evolutional period of the DEAF becomes small, the maximal value of the DEAF reduces and the sustainment time of the maximal DEAF becomes shorter. When the nonlinear interaction of the Kerr medium with the field is strong enough ($\chi/g = 5$), we find that the DEAF tends to zero in the overall time evolution process (see Fig. 2(c)). This result corresponds with the fact that in the strong nonlinear interaction of the Kerr medium with the field, the field and the two atoms are almost decoupled.

4 The Quantum Entanglement Between the Two Two-Level Atoms

The above results show that the effective Hamiltonian given by (1) will lead to the entanglement between the two atoms and the coherent field. Therefore, the state of the two atoms

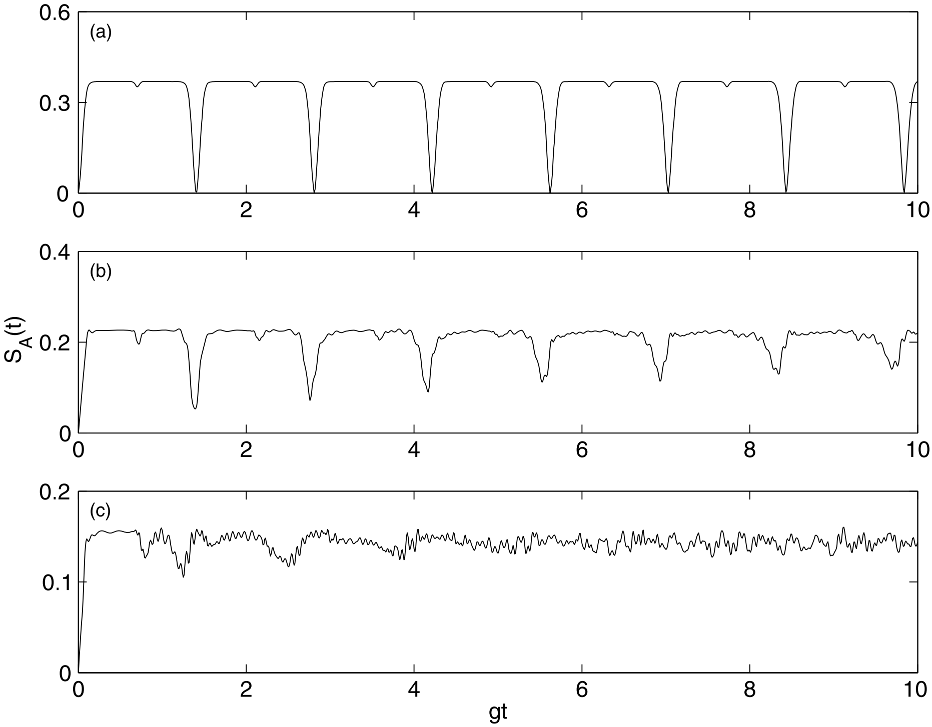

Fig. 1 The influence of the atomic dipole-dipole interaction on the properties of the entanglement between the two atoms and the coherent field, for $N=10$, $\chi/g=1$, (**a**) $\Omega/g=0$; (**b**) $\Omega/g=25$; (**c**) $\Omega/g=50$

evolves into a mixed state. In this case, the degree of the entanglement between the two atoms cannot be measured by the reduced entropy. However, the relative entropy of the entanglement is a good measure for the degree of the entanglement of two-particle mixed state, which is defined as [17]

$$E_R(\rho)=\min_{\sigma\in D} S(\rho\parallel\sigma). \tag{17}$$

Where $S(\rho\parallel\sigma)=\mathrm{Tr}[\rho(\ln\rho-\ln\sigma)]$ is the relative quantum entropy, its minimum is taken over D, the set of all disentangled states. The relative entropy of the entanglement is viewed as the minimal 'distance' between the state ρ and the disentangled state σ. For a pure state, the relative entropy of the entanglement reduces to its reduced entropy. While for mixed states, it is usually difficult to calculate the relative entropy of the entanglement except for some specific states. Recently, the following theorem about the relative entropy of the entanglement has been proven [15], and it is considerably suitable for our analysis.

If a bipartite quantum state can be expressed as

$$\rho=\sum_{n,m} a_{n,m}|\psi_n\phi_n\rangle\langle\psi_m\phi_m|. \tag{18}$$

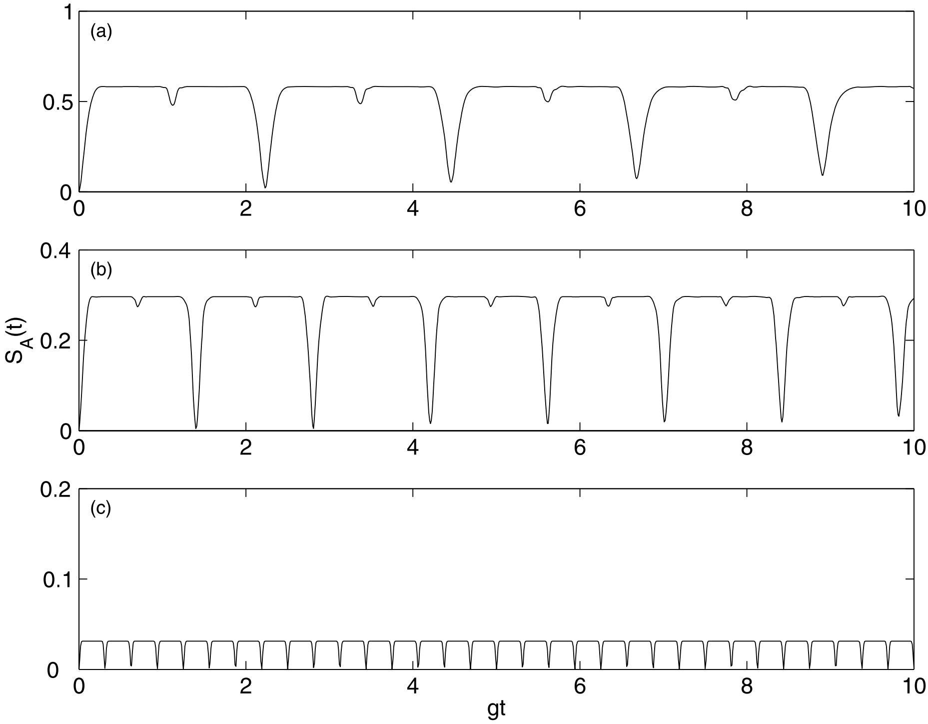

Fig. 2 The influence of the nonlinear interaction of the Kerr medium with the field on the properties of the entanglement between the two atoms and the coherent field, for $N = 10$, $\Omega/g = 10$, (**a**) $\chi/g = 0.5$; (**b**) $\chi/g = 1$; (**c**) $\chi/g = 5$

Then the relative entropy of the entanglement is given by

$$E_R(\rho) = -\sum_n a_{n,n} \ln a_{n,n} + \mathrm{Tr}(\rho \ln \rho), \tag{19}$$

and the disentangled state σ that minimizes the relative quantum entropy is given by

$$\sigma = \sum_n a_{n,n} \ln a_{n,n} + \mathrm{Tr}(\rho \ln \rho). \tag{20}$$

It is clear that $\rho_A(t)$ given by (9) could take the form

$$\rho_A(t) = \sum_{n,m} a_{n,m} |n, n\rangle\langle m, m|, \tag{21}$$

where $a_{n,m} = \langle n, n|\rho_A(t)|m, m\rangle$. Hence, the relative entropy of the entanglement for the state in (9) is written as

$$E_R(\rho_A) = -\sum_m a_{m,m} \ln a_{m,m} - S(\rho_A) = -\sum_m a_{m,m} \ln a_{m,m} + \sum_{i=1}^{4} \lambda_A^i \ln \lambda_A^i. \tag{22}$$

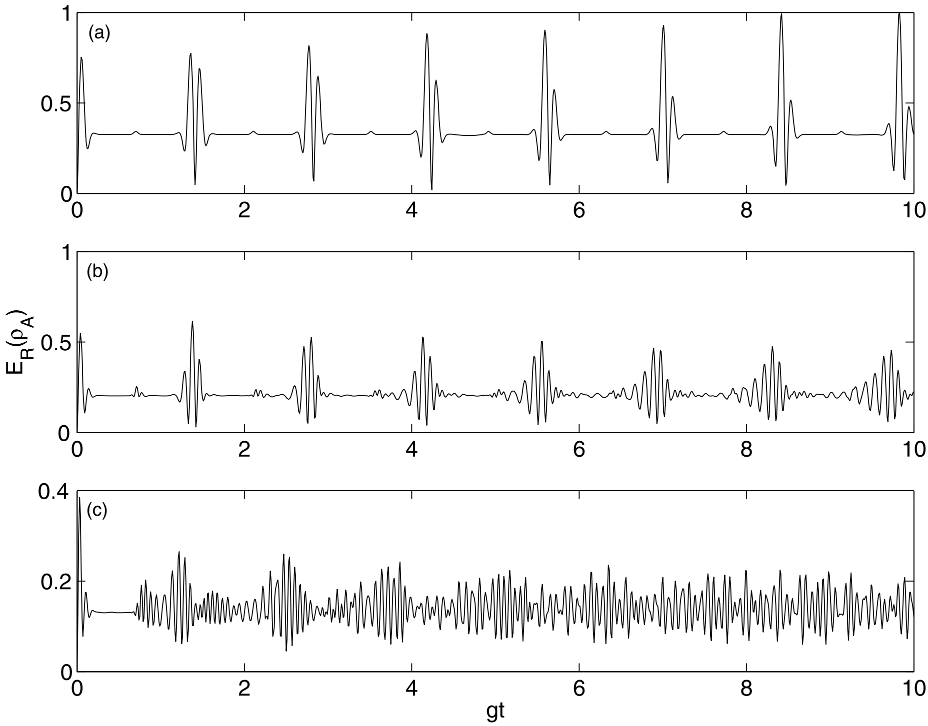

Fig. 3 The influence of the atomic dipole-dipole interaction on the properties of the entanglement between the two atoms, for $N = 10$, $\chi/g = 1$, (**a**) $\Omega/g = 0$; (**b**) $\Omega/g = 25$; (**c**) $\Omega/g = 50$

Therefore, $E_R(\rho_A)$ reflects the degree of the entanglement between the two atoms (DEAA). The numerical results of (22) are shown in Figs. 3 and 4 and the parameters used are the same as in Figs. 1 and 2.

Figure 3 displays the numerical results for the time evolution of the DEAA for $N = 10$, $\chi/g = 1$ and the different atomic dipole-dipole interaction intensities ($\Omega/g = 0, 25, 50$). When the atomic dipole-dipole interaction is equal to zero ($\Omega/g = 0$), the time evolution of the DEAA exhibits periodic oscillation. With the intensity of the atomic dipole-dipole interaction increasing, the time evolution of the DEAA exhibits irregular oscillation and the average value of the DEAA gradually decreases.

Figure 4 displays the numerical results for the time evolution of the DEAA for $N = 10$, $\Omega/g = 10$ and the different nonlinear interaction of the Kerr medium with the field ($\chi/g = 0.5, 1, 5$). From the Figs. 4(a–c), it is observed that the DEAA evolves periodically. With the increase of the nonlinear interaction of the Kerr medium with the field, the evolutional period of the DEAA becomes small, the stable value of the DEAA reduces and the sustainment time of the stable DEAA becomes shorter. When the nonlinear interaction of the Kerr medium with the field is strong enough ($\chi/g = 5$), we find that the DEAA tends to zero in the overall time evolution process (see Fig. 4(c)). This result corresponds with the fact that in the strong nonlinear interaction of the Kerr medium with the field, the two atoms are almost decoupled.

Comparing Fig. 1 with Fig. 3 and Fig. 2 with Fig. 4, we can find that the time evolution of the DEAA is approximately opposite to the time evolution of the DEAF, that is, when the

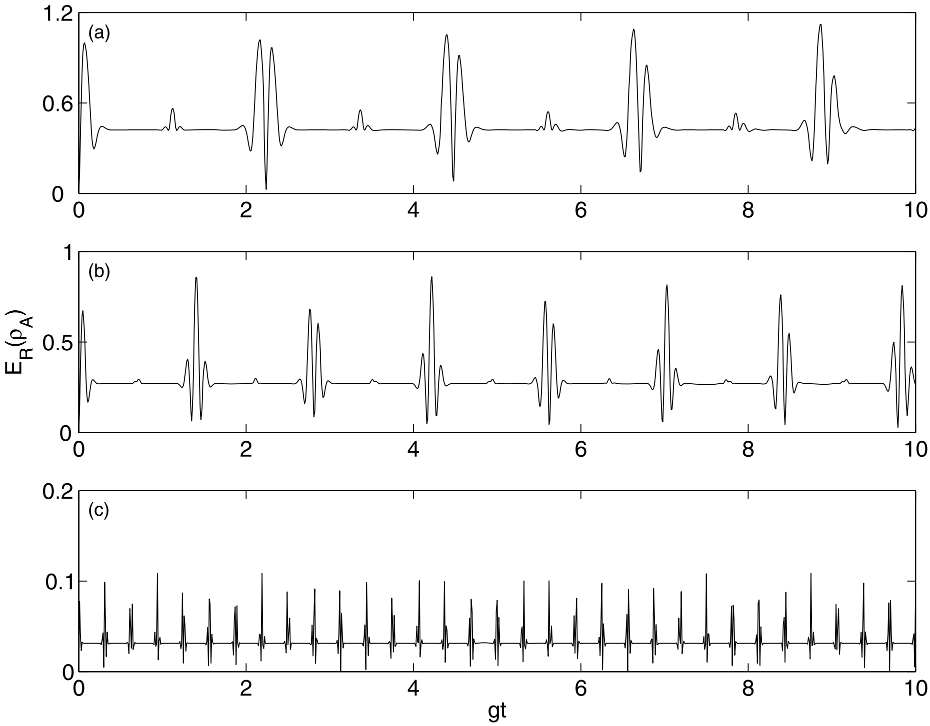

Fig. 4 The influence of the nonlinear interaction of the Kerr medium with the field on the properties of the entanglement between the two atoms, for $N = 10$, $\Omega/g = 10$, (**a**) $\chi/g = 0.5$; (**b**) $\chi/g = 1$; (**c**) $\chi/g = 5$

latter increases with time, the former decreases with time. This comes from the fact that the DEAFcan impair the DEAA due to the two atoms interacting with the coherent field.

5 Conclusions

In this paper, we has studied the properties of quantum entanglement in the two-photon Tavis–Cummings model with a Kerr nonlinearity and examined the influences of the nonlinear interaction of the Kerr medium with the field and the atomic dipole-dipole interaction on the properties of quantum entanglement of the system. The results can be concluded as follows: First, the addition of a Kerr medium to the two-photon Tavis–Cummings model has an important effect on the properties of quantum entanglement in the system. When the nonlinear interaction of the Kerr medium with the field is very weak, it leads to an increase of the sustainment time of the maximal (or stable) degree of the entanglement. When the nonlinear interaction of the Kerr medium with the field is very strong, it results in a decrease of the degree of the entanglement and the two atoms and the field are decoupled, the two atoms are decoupled. Second, the atomic dipole-dipole interaction leads to the decrease of the degrees of the entanglement and the irregular oscillation of the time evolution of the entanglement. Finally, the time evolution of the degree of the entanglement between the two atoms and the field is approximately opposite to the time evolution of the degree of the entanglement between the two atoms, that is, the degree of the entanglement between the two atoms and the field can impair the degree of the entanglement between the two atoms. Our results are important for the experimental realization of the preparation of entangled states.

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